Line bundles on Curves

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1 Introduction

In this talk, we will introduce the notion of a divisor on a curve. Especially we will discuss about the relationship between a divisor and a line bundle on a curve. Then we will briefly introduce so-called Čech cohomology of a quasi-coherent sheaf for a "nice" scheme. After that we will show that the dimension of the global sections of a line bundle can be described by the dimension of the 1st Čech cohomology and the degree of the line bundle for a curve. This is a weak form of the Riemann-Roch theorem.

2 Effective Cartier Divisor

Now recall the definition of an effective cartier divisor on a scheme. Let $D \subset X$ be a closed subscheme of a scheme X. Let $\mathcal{I}_D$ the corresponding quasi-coherent ideal sheaf of D. We call D an **effective cartier divisor** if $\mathcal{I}_D$ is invertible. We define $\mathcal{O}_X(D) = \mathcal{I}_D^\vee = \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X)$. Clearly this is invertible. We often say the corresponding line bundle as an effective cartier divisor.

Remark 2.1. Let D be an effective cartier divisor. Then $0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ is a short exact sequence.

Example 2.2. Let A be a dedekind domain and let $X = \operatorname{Spec} A$. Then for any ideal $I \subset A$, I is invertible by the definition of a dedekind domain. Hence $D = V(I)$ is an effective cartier divisor for any I. This shows that any closed subscheme of X is an effective cartier divisor.

We want to define a regular section of an invertible sheaf on X.

Definition 2.3. Let $\mathcal{L}$ be an invertible sheaf on a scheme X. A global section $s \in \Gamma(X, \mathcal{L})$ is **regular** if $\mathcal{O}_X \to \mathcal{L}$ by $f \mapsto fs$ is injective.

Also we define a closed subscheme cut out by $s = 0$ denoted as V(s) where $s \in \Gamma(X, \mathcal{L})$ is a global section of a line bundle $\mathcal{L}$.

Definition 2.4. Define $\mathcal{I} = \operatorname{Im}(\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \xrightarrow{s^\vee} \mathcal{O}_X)$ where the morphism $s^\vee$ is defined by $f \mapsto f(s)$. Then $\mathcal{I}$ is a quasi-coherent ideal sheaf of X since $QCoh_X$ is an abelian category. $V(s)$ is the closed subscheme of X defined by the ideal sheaf $\mathcal{I}$.

We are ready to introduce the following proposition. This shows that an effective cartier divisor is just a line bundle with a fixed regular section paired.

Proposition 2.5. *Let X be a scheme and let $\mathcal{L}$ be a line bundle of X.*

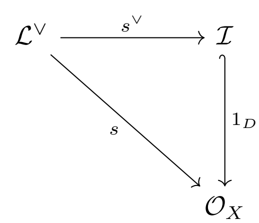
$$\{\text{Effective cartier divisors on } X\} \leftrightarrow \{(\mathcal{L}, s) \text{ where } s \text{ is a regular section of } \mathcal{L}\}/\sim$$
$$D \mapsto (\mathcal{O}_X(D), 1_D) \text{ where } 1_D : \mathcal{I}_D \hookrightarrow \mathcal{O}_X$$
$$V(s) \leftarrow\!\shortmid (\mathcal{L}, s)$$

Here $(\mathcal{L}, s) \sim (\mathcal{L}', s')$ if the line bundles are isomorphic and the regular sections send to each other by the isomorphism.

Proof. We first show that 1_D is regular for an effective cartier divisor D. We are enough to show that $\mathcal{O}_X(U) \xrightarrow{\cdot 1_D(U)} \Gamma(U, \mathcal{O}_X(D))$ is injective for any affine $U = \operatorname{Spec} A \subset X$. This is because if $1_D(U)$ is injective for all affine $U \subset X$, then since localisation is an exact functor, it preserves injectivity and hence 1_D is stalk-wise injective.

Since $\mathcal{I}_D$ is quasi-coherent and U is affine, $\Gamma(U, \mathcal{O}_X(D)) = \mathcal{H}om(\mathcal{I}_D|_U, \mathcal{O}_X|_U) = \mathcal{H}om(\widetilde{\mathcal{I}_D(U)}, \widetilde{A}) \cong \operatorname{Hom}_A(\mathcal{I}_D(U), A)$. Let $\mathcal{I}_D(U) = I \subset A$. Then $\mathcal{O}_X(U) \xrightarrow{\cdot 1_D(U)} \Gamma(U, \mathcal{O}_X(D))$ becomes $A \to \operatorname{Hom}_A(I, A)$ by $a \mapsto (\phi_a : i \mapsto ai)$. Note that here I is an invertible A-module. Let $\phi_a = \phi_b$. Then for all $i \in I$, $ai = bi$. Since I is invertible, there exists an invertible A-module M such that $I \otimes_A M \cong A$. Then $\sum i_k \otimes m_k = 1$ for a finite index set $k \in K$. $a = a \sum i_k \otimes m_k = \sum a i_k \otimes m_k = \sum b i_k \otimes m_k = b \sum i_k \otimes m_k = b$. Hence $1_D(U)$ is injective. We conclude that 1_D is regular.

Now we want to prove that $V(s)$ is indeed an effective cartier divisor i.e., the corresponding ideal sheaf $\mathcal{I}$ is invertible. Since s is regular, $0 \to \mathcal{O}_X \xrightarrow{\cdot s} \mathcal{L}$ is exact. Tensoring a line bundle is exact (because it's locally free) hence if we tensor $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, we get $0 \to \mathcal{L}^\vee \xrightarrow{\cdot s \otimes \mathcal{L}^\vee} \mathcal{O}_X$ exact. Note that $s^\vee = \cdot s \otimes \mathcal{L}^\vee$. This implies that $s^\vee$ is injective and surjective to it's image and hence $\mathcal{L}^\vee \cong \mathcal{I}$. This holds because we are dealing with a category of quasi-coherent sheaves. This concludes that $V(S)$ is an effective cartier divisor, say D. By the isomorphism above, $\mathcal{H}om(\mathcal{I}, \mathcal{O}_X) \xrightarrow{\mathcal{H}om(\cdot s \otimes \mathcal{L}^\vee, \mathcal{O}_X)} \mathcal{H}om(\mathcal{L}^\vee, \mathcal{O}_X)$ is also an isomorphism[1] and hence $\mathcal{O}_X(D) \cong \mathcal{L}$. Also we can check that the isomorphism sends 1_D to s by the given commutative diagram.



Hence $(\mathcal{L}, s) \mapsto D \mapsto (\mathcal{O}_X(D), 1_D) \sim (\mathcal{L}, s)$ holds.

For the opposite way, $D \mapsto (\mathcal{O}_X(D), 1_D) \mapsto V(1_D) = D$ by the definition. □

3 Divisors on a projective normal curve

Recall that a curve is a 1-dimensional integral separated finite type scheme over a field k. From now on, we will only focus on a projective normal curve C over k where $k = \overline{k}$. We define what a divisor is on a curve.

Definition 3.1. A **divisor** $D \in \operatorname{Div}(C)$ on a curve C is a formal sum $D = \sum_{x \in C} n_x[x]$ where $x \in C$ are closed and n_x are zero for all but finitely many $x \in C$. D is **effective** if $n_x \geq 0$ for all $x \in C$.

We define $\mathcal{O}_C(D)$ by $\Gamma(U, \mathcal{O}_C(D)) := \{f \in K \mid \forall x \in U \text{ closed}, v_x(f) \geq -n_x\}$. Here v_x is the valuation of the discrete valuation ring $\mathcal{O}_{C,x}$ and K is the function field of C. $\mathcal{O}_C(D)$ is invertible for any divisor D. Read Scholze's Algebraic Geometry I note 104pg for a proof.

Proposition 3.2. *Let $D, D' \in \operatorname{Div}(C)$. Then $\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \cong \mathcal{O}_C(D + D')$.*

Proof. Consider the following morphism.

$$\Gamma(U, \mathcal{O}_C(D)) \otimes_{\Gamma(U, \mathcal{O}_C)} \Gamma(U, \mathcal{O}_C(D')) \to \Gamma(U, \mathcal{O}_C(D + D'))$$

by $f \otimes g \mapsto fg$ where $D = \sum n_x[x]$, $D' = \sum n'_x[x]$, $\forall x \in U, v_x(f) \geq -n_x$, $v_x(g) \geq -n'_x$.

[1] $\mathcal{H}om(-, \mathcal{F})$ is generally only left exact but if $\mathcal{F} = \mathcal{O}_X$, the functor is exact.

Then $fg \in \{h \in K \mid \forall x \in U, v_x(h) \geq -n_x - n'_x\} = \Gamma(U, \mathcal{O}_C(D + D'))$. The sheaf morphism $\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \to \mathcal{O}_C(D+D')$ can be defined by this. Recall that a sheaf morphism is an isomorphism if it's stalk-wise isomorphic. Let $x \in C$ be a closed point. $\mathcal{O}_C(D)_x = \omega^{-n_x} \cdot \mathcal{O}_{C,x}$ for the uniformizer ω of $\mathcal{O}_{C,x}$. This is because $v_x(\omega) = 1$. Then we can get that $(\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D'))_x = \mathcal{O}_C(D)_x \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_C(D')_x \cong (\mathcal{O}_C(D + D'))_x$. □

Remark 3.3. Direct corollary of this proposition is that $\mathcal{O}_C(D)^\vee \cong \mathcal{O}_C(-D)$.

Corollary 3.4. *Let $D \in \mathrm{Div}(C)$ be an effective divisor. Then $\mathcal{O}_C(D)$ is an effective cartier divisor.*

Proof. Let $D = \sum n_x[x]$ and $n_x \geq 0$ for all $x \in C$. Consider $\mathcal{O}_C(-D)$. Then since $v_x(f) \geq 0$ for all x, $\mathcal{O}_C(-D) = \mathcal{I}$ is an invertible ideal sheaf of $\mathcal{O}_C$. Hence $\mathcal{O}_C(D)$ is an effective cartier divisor. □

Remark 3.5. By repeating **Prop 3.2.**, for $D = \sum n_x[x]$, $\mathcal{O}_C(D) \cong \bigotimes \mathcal{O}_C(x)^{\otimes n_x}$. Here $\mathcal{O}_C(D)^{\otimes -1} = \mathcal{O}_C(D)^\vee$

We define $\deg D = \sum n_x$. After introducing Čech cohomology, degree of D will play an important role by the Riemann-Roch theorem.

4 Introduction to Čech Cohomology

Now let us fix the scheme X be a variety over $k = \overline{k}$. Recall that a variety is a finite type separated integral scheme over k. We fix some notations as below.
$X = \bigcup_{i=1}^r$ be an affine cover. $J \subset \{0, \dots, r\} = I$, $U_J = \bigcap_{j \in J} U_j$. $\mathcal{F}$ is a quasi-coherent sheaf on X.

Since X is separated, U_J is affine. Now we define a Čech complex denoted by $C^\bullet(\mathcal{F})$.

Definition 4.1.

$$0 \to C^0(\mathcal{F}) \to \cdots \to C^p(\mathcal{F}) \xrightarrow{d^p} C^{p+1}(\mathcal{F}) \to \cdots$$

is called a **Čech Complex** where $C^p(\mathcal{F}) = \bigoplus_{J \subset I, |J| = p+1} \mathcal{F}(U_J)$ and $d^p : (-1)^n \mathrm{res}^{\mathcal{F}(U_J)}_{\mathcal{F}(U_{J \cup \{j\}})}$ and $n+1$ is the position of j in $J \cup \{j\}$ counted from 0. If there is no such restriction, we send to 0. Note that $C^p(\mathcal{F})$ are k-vector spaces.

Example 4.2. Let $X = U_0 \cup U_1$. Then

$$0 = C^{-1}(\mathcal{F}) \to C^0(\mathcal{F}) = \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \to C^1(\mathcal{F}) = \mathcal{F}(U_0 \cap U_1) \to C^2(\mathcal{F}) = 0$$

Let $(a_0, a_1) \in \mathcal{F}(U_0) \oplus \mathcal{F}(U_1)$. Then $d^0(a_0, a_1) = a_0|_{U_0 \cap U_1} - a_1|_{U_0 \cap U_1}$.

Remark 4.3. By a direct computation, we can check that $d^{p+1} \circ d^p = 0$. Hence $C^\bullet$ is indeed a complex.

Now we define a cohomology from this complex.

Definition 4.4. Let X and $\mathcal{F}$ be given as above. The p-th cohomology of $\mathcal{F}$ on X is $H^p(X, \mathcal{F}) = \mathrm{Ker}\, d^p / \mathrm{Im}\, d^{p-1}$. If $\mathcal{F}$ is coherent, the dimension of p-th cohomology as a k-vector space is denoted by $\dim H^p(X, \mathcal{F}) = h^p(X, \mathcal{F})$.

Remark 4.5. This is independent of choice of affine covers but we will skip this. See Vakil's note **Theorem 18.2.2** for a proof.

We introduce some basic properties of Čech cohomology.

Proposition 4.6. *The following properties hold.*

(a) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

(b) If X is affine, $H^p(X,\mathcal{F}) = 0$ for all $p > 0$.

(c) Let X be projective. Then for all $p > \dim X$, $H^p(X,\mathcal{F}) = 0$.

(d) Let $i : X \hookrightarrow Y$ be a closed immersion. Then $H^p(Y, i_*\mathcal{F}) = H^p(X,\mathcal{F})$ for all p.

Proof. (a): Consider the following complex $0 \to C^0(\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{F}) \to \cdots$. Since $\operatorname{Im} d^{-1} = 0$, $H^0(X,\mathcal{F}) = \operatorname{Ker} d^0$. Let $\phi \in C^0(\mathcal{F})$. Then $\phi = (\phi_1, \cdots, \phi_r)$ where $\phi_i \in \mathcal{F}(U_i)$. Then $(d^0\phi)_{ij} = (\phi_i - \phi_j)|_{U_i \cap U_j} = 0$ which is the sheaf property of global sections. Hence $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

(b): This is trivial because X itself is an affine cover of X and $C^1(\mathcal{F}) = 0$.

(c): Let $\dim X = n$. Since X is projective, X is a closed subscheme of $\mathbb{P}^r_k$ for some r. Hence we can find $f_0, \ldots, f_n$ such that $V_+(f_0, \ldots, f_n) \cap X = \emptyset$. This is possible because $f_0, \ldots, f_n$ reduces dimension $n+1$. Now define $U_i = X \setminus V_+(f_i) = D_+(f_i) \cap X$. Then $\bigcup U_i = X$. Here U_i is affine because $D_+(f_i)$ is affine and $D_+(f_i) \cap X$ is a closed subscheme of affine which is also affine. Since $C^p(\mathcal{F}) = 0$ for all $p > n$.

(d): Let $\{U_1, \ldots, U_r\}$ covers Y. Then $\{i^{-1}(U_1)), \ldots, i^{-1}(U_r)\}$ covers X. Now since $i_*\mathcal{F}(U_J) = \mathcal{F}(i^{-1}(U_J))$, $C^p(i_*\mathcal{F}) = C^p(\mathcal{F})$. ☐

Proposition 4.7. *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence. Then there exists a long exact sequence induced by the short exact sequence namely*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}_1) & \longrightarrow & \Gamma(X,\mathcal{F}_2) & \longrightarrow & \Gamma(X,\mathcal{F}_3) \\ & \swarrow & & & & & \\ & & H^1(X,\mathcal{F}_1) & \longrightarrow & H^1(X,\mathcal{F}_2) & \longrightarrow & H^1(X,\mathcal{F}_3) \\ & \swarrow & & & & & \\ & & H^2(X,\mathcal{F}_1) & \longrightarrow & \cdots . & & \end{array}$$

Proof. (Sketch) We can check that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^p(\mathcal{F}_1) & \longrightarrow & C^p(X,\mathcal{F}_2) & \longrightarrow & C^p(X,\mathcal{F}_3) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^p} & & \downarrow{\scriptstyle d^p} & & \downarrow{\scriptstyle d^p} & & \\ 0 & \longrightarrow & C^{p+1}(\mathcal{F}_1) & \longrightarrow & C^{p+1}(X,\mathcal{F}_2) & \longrightarrow & C^{p+1}(X,\mathcal{F}_3) & \longrightarrow & 0 \end{array}$$

Rows are exact because $\mathcal{F}$ is quasi-coherent, all U_J are affine, and direct sum is exact. Here we use the fact that taking global sections is exact for affine schemes. By applying snake's lemma, we can get the long exact sequence. ☐

Remark 4.8. Taking global sections as a functor is left exact in general. Defining Čech cohomology shows that this left exactness of global sections can be extended to a long exact sequence.

Now we again assume X to be projective.

Definition 4.9. Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. The **Euler characteristic** of $\mathcal{F}$ denoted by $\chi(X,\mathcal{F}) = \sum_{n=0}^{\dim X} (-1)^n h^n(X,\mathcal{F})$.

Proposition 4.10. *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent sheaves. Then $\chi(X,\mathcal{F}_2) = \chi(X,\mathcal{F}_1) + \chi(X,\mathcal{F}_3)$.*

Proof. (Sketch) Consider the long exact sequence. Then use the rank-nullity theorem. (If V_i are k-vector spaces and $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ exact, then $\sum(-1)^i \dim V_i = 0$) □

5 Back to curves and Riemann-Roch Theorem

Let us go back to the situation of curves. Let C be a projective normal curve as before and $\mathcal{F}$ be a coherent sheaf on C. Since C is 1-dimensional, $h^p(C, \mathcal{F}) = 0$ for all $p > 1$. Hence $\chi(C, \mathcal{F}) = \dim \Gamma(X, \mathcal{F}) - \dim H^1(C, \mathcal{F})$.

Definition 5.1. Let $\mathcal{L}$ be a line bundle on C. The **degree** of $\mathcal{L}$ denoted by $\deg \mathcal{L}$ is defined by $\deg \mathcal{L} = \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C))$.

Theorem 5.2 (Weak version of the Riemann-Roch theorem)**.** *Let D be a divisor on C, $\mathcal{L}$ be a line bundle on C such that $\mathcal{L} \cong \mathcal{O}_C(D)$. Then $\deg \mathcal{L} = \deg D$.*
In other words, $\dim \Gamma(C, \mathcal{L}) = \dim H^1(C, \mathcal{L}) + \deg D + \chi(C, \mathcal{O}_C)$.

Proof. Recall that we can make a short exact sequence from an effective cartier divisor. Let $p \in C$ be a closed point. Then p is an effective cartier divisor. Hence $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$ is exact. Here $\mathcal{O}_p$ is not a stalk at p but a skyscraper sheaf at p. If we tensor with $\mathcal{L}$, we get

$$0 \to \mathcal{O}_C(-p) \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{O}_p \otimes \mathcal{L} \to 0.$$

By the additive law of Euler characteristics, $\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C(-p) \otimes \mathcal{L}) + \chi(C, \mathcal{O}_p \otimes \mathcal{L})$.
From $\chi(C, \mathcal{O}_p \otimes \mathcal{L}) = h^0(C, \mathcal{O}_p \otimes \mathcal{L}) - h^1(C, \mathcal{O}_p \otimes \mathcal{L})$, $h^1(C, \mathcal{O}_p \otimes \mathcal{L}) = h^1(p, \mathcal{O}_p \otimes \mathcal{L}) = 0$ since p is affine. For $h^0(C, \mathcal{O}_p \otimes \mathcal{L})$, $\mathcal{O}_p \otimes \mathcal{L} \cong \mathcal{O}_p$ and hence $h^0(C, \mathcal{O}_p \otimes \mathcal{L}) = h^0(C, \mathcal{O}_p) = h^0(p, \mathcal{O}_p)$. Then $\Gamma(p, \mathcal{O}_p) = \kappa(p)$ where $\kappa(p)$ is a finite extension of k. In our assumption, k is algebraically closed and hence $h^0(p, \mathcal{O}_p \otimes \mathcal{L}) = 1$. This implies that $\chi(C, \mathcal{L}) = \chi(C, \mathcal{O}_C(-p) \otimes \mathcal{L}) + 1$. Replace $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{O}_C(p)$. Then we get, $\chi(\mathcal{L}) + 1 = \chi(\mathcal{O}_C(p) \otimes \mathcal{L})$. This implies that $\deg \mathcal{L} = \chi(\mathcal{O}_C(D)) - \chi(\mathcal{O}_C) = \chi(\bigotimes \mathcal{O}_C(x)^{\otimes n_x}) - \chi(\mathcal{O}_C) = \sum n_x = \deg D$ by repeating the above equation. □

Remark 5.3. Riemann showed that $\dim \Gamma(C, \mathcal{L}) \geq \deg D + \chi(C, \mathcal{O}_C)$ and Roch filled the missing gap.

Definition 5.4. The **genus** g of the curve C is $g = 1 - \chi(\mathcal{O}_C)$.

Corollary 5.5. *Let C be a curve as above and let the genus of the curve be g. Let $\mathcal{L}$ be a line bundle on a curve C such that $\deg \mathcal{L} = g$. Then there exists an effective cartier divisor D such that $\mathcal{L} \cong \mathcal{O}_C(D)$.*

Proof. $\chi(\mathcal{L}) - \chi(\mathcal{O}_C) = g$ implies that $\chi(\mathcal{L}) = 1$. Hence $\dim \Gamma(C, \mathcal{L}) = \dim H^1(C, \mathcal{L}) + 1 > 0$. Since $\Gamma(C, \mathcal{L})$ is non-zero, we can pick a non-zero global section $s \in \Gamma(C, \mathcal{L})$. We claim that s is regular. We are enough to show that $s|_U \neq 0$ for all open $U \subset C$. Since C is integral, for any $x \in U$, $s|_U \mapsto s_x \neq 0$ is injective. Hence $s|_U \neq 0$ and s is regular. Then $V(s) = D$ is an effective cartier divisor and hence by **Prop 2.5.** $\mathcal{O}_C(D) \cong \mathcal{L}$. □